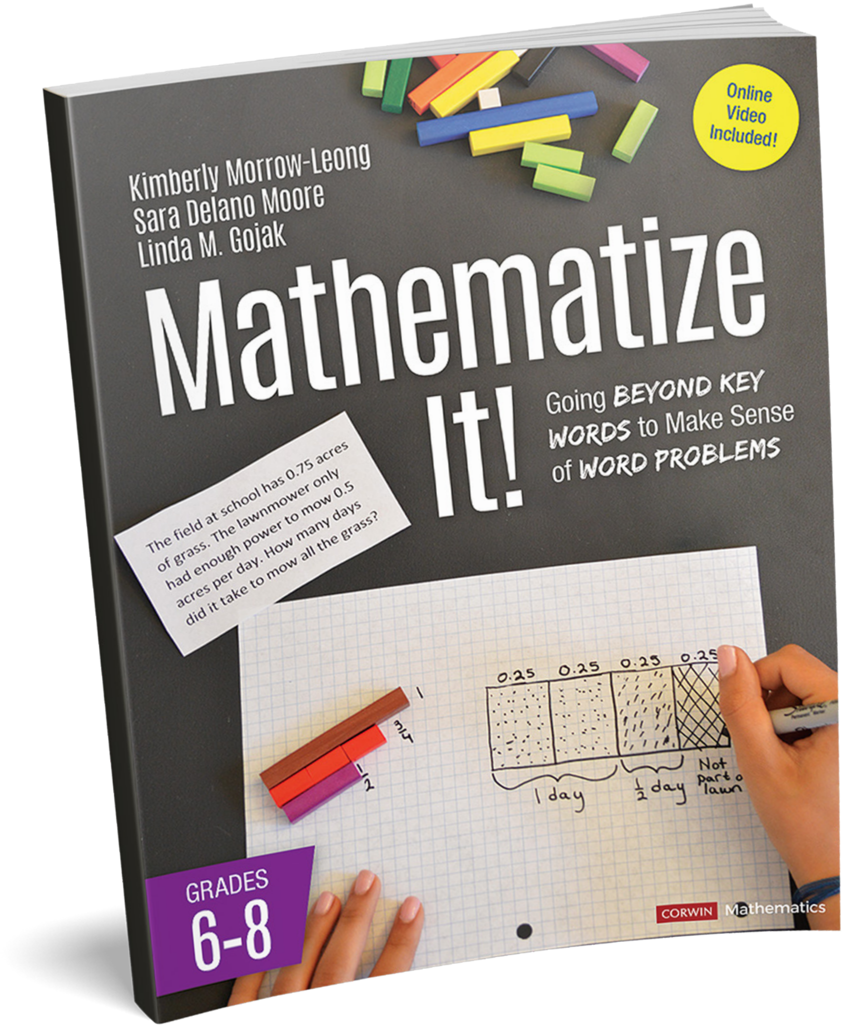

Thank you

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Please enjoy this complimentary excerpt from *Mathematize It! Grades 6-8* by Kimberly Morrow-Leong, Sara Delano Moore, and Linda M. Gojak.

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CHAPTER ONE

Introduction

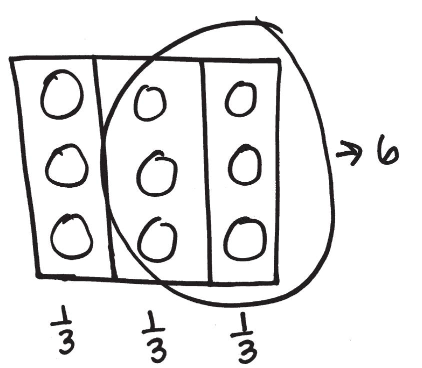
Why You Need to Teach Students to Mathematize

Imagine you are a new teacher. You are teaching eighth grade at a new school and are eager to get to know your students—their interests, skills, and how prepared they are to meet the challenges of eighth grade. You have just emerged from your teacher education program knowing various approaches you have seen modeled in classrooms and described in the literature, some of which you have tried with varying degrees of success. You aren't sure what approaches you want to use but are excited about challenging your students, introducing the rigor you have read so much about. But first, you need to know what your students can and can't do.

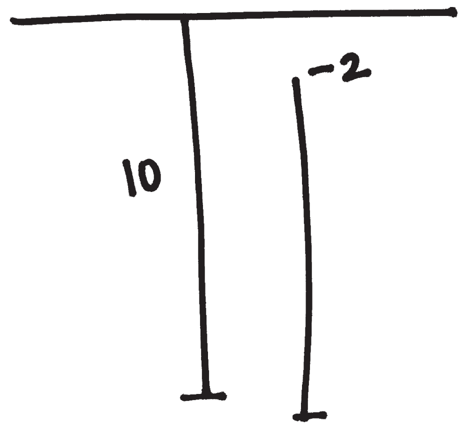
You decide to start with a couple of word problems, ones that involve relatively simple mathematical operations:

> *Mrs. King wanted her American history students to do a project about the Emancipation Proclamation. $\frac{2}{3}$ of the class chose to make podcasts. The other 9 students chose to create graphic novels. How many students are in Mrs. King's American history class?*
>
> *Armando started his descent into the cave. He was 10 feet down before he realized that he had forgotten to bring a flashlight. He climbed back up to the 2-foot mark to take the flashlight his friend handed to him. How many feet did he have to climb to get the flashlight?*

You circulate around the room, noting who draws pictures, who writes equations, and who uses the manipulatives you have put at the center of the table groups. While some students take their time, quite a few move quickly. Their hands go up, indicating they have solved the problems. As you check their work, one by one, you notice most of them got the first problem wrong, writing the equation $\frac{2}{3}\times 9=6$. Some even include a sentence saying, "6 students will do a podcast." Only one student in this group draws a picture. It looks like this:



Even though the second problem demands an understanding of integers, a potentially complicating feature, most of these same students arrive at the correct answer, despite the fact that they do not write a correct equation to go with it. They write the incorrect equation 10 – 2 = 8 and are generally able to find the correct answer of +8, representing an 8-foot climb toward the cave opening. Some write K C C above their equation. You notice that other students make a drawing to help them solve this problem. Their work looks something like this:



To learn more about how your students went wrong with the history assignment problem, you call them to your desk one by one and ask about their thinking. A pattern emerges quickly. All the students you talk to zeroed in on two key elements of the problem: (1) the portion of students who did a podcast ($\frac{2}{3}$) and (2) the word "of". One student tells you, "*Of* always means to multiply. I learned that a long time ago." Clearly, she wasn't the only student who read the word *of* and assumed she had to multiply by the only other number given in the problem. While a key word strategy led students astray in the first problem, visualizing the problem situation in the second problem led students to a correct answer, even if they were not able to write an accurate equation for the problem situation.

Problem-Solving Strategies Gone Wrong

In our work with teachers, we often see students being taught a list of "key words" that are linked to specific operations. Students are told, "Find the key word and you will know whether to add, subtract, multiply, or divide." Charts of key words often hang on classroom walls, even in middle school. Key words are a strategy that works often enough that teachers continue to rely on them. They also seem to work well enough that *students* continue to rely on them. But as we saw in the history assignment problem, not only are key words not enough to solve a problem, they can also easily lead students to an incorrect operation or to an operation involving two numbers that aren't related (Karp, Bush, & Dougherty, 2014). As the history assignment problem reveals, different operations could successfully be called upon, depending on how the student approaches the problem:

1. A student could use subtraction $\left(1-\frac{2}{3}\right)$ to determine that $\frac{1}{3}$ of the students in the class made graphic novels.
2. A student could use division to find the number of students in the class, dividing the 9 students doing graphic novels by $\frac{1}{3}$ of the class to get 27, the number of students in the whole class. This could even be modeled using an array solution strategy like the one in the student's drawing seen earlier.

Let's return to your imaginary classroom. Having seen firsthand the limitations of key words—a strategy you had considered using—where do you begin? What instructional approach would you use? One of the students mentioned a strategy she likes called CUBES. If she learned it from an elementary teacher and still uses it, you wonder if it has value. Your student explains that CUBES has these steps:

Circle the numbers

Underline important information

Box the question

Eliminate unnecessary information

Solve and check

She tells you that her teachers walked students through the CUBES protocol using a "think-aloud" for word problems, sharing how they used the process to figure out what is important in the problem. That evening, as you settle down to plan, you decide to walk through some problems like the history assignment problem using CUBES. Circling the numbers is easy enough. You circle $\frac{2}{3}$ (podcast) and 9 (students), wondering briefly what students might do with the question "How many?" Perhaps it's too early to think of that for now.

Then you tackle "important information." What is important here in this problem? Maybe the fact that there are two different assignments. Certainly it's important to recognize that students do one of two kinds of history assignment. You box the question, but unfortunately the question doesn't help students connect $\frac{2}{3}$ to 9 with a single operation.

If you think this procedure has promise as a way to guide students through an initial reading of the problem, but leaves out how to help students develop a genuine understanding of the problem, you would be correct.

What is missing from procedural strategies such as CUBES and strategies such as key words, is—in a word—*mathematics* and the understanding of where it lives within the situation the problem is presenting. Rather than helping students learn and practice quick ways to enter a problem, we need to focus our instruction on helping them develop a deep understanding of the mathematical principles behind the operations and how they are expressed in the problem. They need to learn to *mathematize*.

Mathematizing: The uniquely human act of modeling reality with the use of mathematical tools and representations.

What Is Mathematizing? Why Is It Important?

Mathematizing is the uniquely human process of constructing meaning in mathematics (from Freudenthal, as cited in Fosnot & Dolk, 2002). Meaning is constructed and expressed by a process of noticing, exploring, explaining, modeling, and convincing others of a mathematical argument. When we teach students to mathematize, we are essentially teaching them to take their initial focus off specific numbers and computations and put their focus squarely on the actions and relationships expressed in the problem, what we will refer to throughout this book as the **problem situation**. At the same time, we are helping students see how these various actions and relationships can be described mathematically and the different operations that can be used to express them. If students understand, for example, that equal-groups multiplication problems, as in the history assignment problem, may include knowing the whole or figuring out the whole from a portion, then they can learn where and how to apply an operator to numbers in the problem, in order to develop an appropriate equation and understand the context. If we look at problems this way, then finding a **solution** involves connecting the problem's context to its general kind of problem situation and to the operations that go with it. The rest of the road to the answer is computation.

Problem situation: The underlying mathematical action or relationship found in a variety of contexts. Often called "problem type" for short.

Solution: A description of the underlying problem situation along with the computational approach (or approaches) to finding an answer to the question.

Operation sense: Knowing and applying the full range of work for mathematical operations (for example, addition, subtraction, multiplication, and division).

Intuitive model of an operation: An intuitive model is "primitive," meaning that it is the earliest and strongest interpretation of what an operation, such as multiplication, can do. An intuitive model may not include all the ways that an operation can be used mathematically.

Making accurate and meaningful connections between different problem situations and the operations that can fully express them requires **operation sense**. Students with a strong operation sense

- Understand and use a wide variety of models of operations beyond the basic and **intuitive models of operations** (Fischbein, Deri, Nello, & Marino, 1985)
- Use appropriate representations of actions or relationships strategically
- Apply their understanding of operations to any quantity, regardless of the class of number
- Can mathematize a situation, translating a contextual understanding into a variety of other mathematical representations

FOCUSING ON OPERATION SENSE

Many of us may assume that we have a strong operation sense. After all, the four operations are the backbone of the mathematics we were taught from day one in elementary school. We know how to add, subtract, multiply, and divide, don't we? Of course we do. But a closer look at current standards reveals nuances and relationships within these operations that many of us may not be aware of, may not fully understand, or may have internalized so well that we don't recognize we are applying an understanding of them every day when we ourselves mathematize problems both in real life and in the context of solving word problems. For example, current standards ask that students develop conceptual understanding and build procedural fluency in four kinds of addition/subtraction problems, including Add-To, Take-From, Compare, and what some call Put Together/Take Apart (we will refer to this category throughout the book as Part-Part-Whole). Multiplication and division have their own unique set of problem types as well. On the surface, the differences between such categories may not seem critical. But we argue that they are. Only by exploring these differences and the relationships they represent can students develop the solid operation sense that will allow them to understand and mathematize word problems and any other problems they are solving, whatever their grade level or the complexity of the problem. It does not mean that students should simply memorize the problem types. Instead they should have experience exploring all the different problem types through word problems and other situations. Operation sense is not simply a means to an end. It has value in helping students naturally come to see the world through a mathematical lens.

USING MATHEMATICAL REPRESENTATIONS

Problem context: The specific setting for a word problem.

Mathematical representation: A depiction of a mathematical situation using one or more of these modes or tools: concrete objects, pictures, mathematical symbols, context, or language.

What would such instruction—instruction aimed at developing operation sense and learning how to mathematize word problems—look like? It would have a number of features. First, it would require that we give students time to focus and explore by doing fewer problems, making the ones they do count. Next, it would facilitate students becoming familiar with various ways to represent actions and relationships presented in a **problem context**. We tend to think of solving word problems as beginning with words and moving toward the use of variables and equations in a neat linear progression. But as most of us know, this isn't how problem solving works. It is an iterative and circular process, where students might try out different representations, including going back and rewording the problem, a process we call telling "the story" of the problem. The model that we offer in this book is based on this kind of active and expanded exploration using a full range of **mathematical representations**. Scholars who study mathematical modeling and problem solving identify five modes of representation: verbal, contextual, concrete, pictorial, and symbolic representations (Lesh, Post, & Behr, 1987).

VERBAL A problem may start with any mode of representation, but a word problem is first presented verbally, typically in written form. After that, verbal representations can serve many uses as students work to understand the actions and relationships in the problem situation. Some examples are restating the problem; thinking aloud; describing the math operations in words rather than symbols; and augmenting and explaining visual and physical representations including graphs, drawings, base 10 blocks, fraction bars, or other concrete items.

CONTEXTUAL The contextual representation is simply the real-life situation that the problem describes. Prepackaged word problems are based on real life, as is the earlier flashlight problem, but alone they are not contextual. Asking students to create their own word problems based on real-life contexts will bring more meaning to the process and will reflect the purposes of mathematics in real life, such as when scientists, business analysts, and meteorologists mathematize contextual information in order to make predictions that benefit us all. This is a process called **mathematical modeling**, which Garfunkel and Montgomery (2019) define as the use of "mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena."

Mathematical modeling: A process that uses mathematics to represent, analyze, make predictions, or otherwise provide insight into real- world phenomena.

CONCRETE Using physical representations such as blocks, concrete objects, and real-world items (for example, money, measuring tools, or items to be measured such as beans, sand, or water), or acting out the problem in various ways, is called **modeling**. Such models often offer the closest and truest representation of the actions and relationships in a problem situation. Even problem situations where negative quantities are referenced can rely on physical models when a feature such as color or position of an object shows that the quantity should be interpreted as negative.

Modeling: Creating a physical representation of a problem situation.

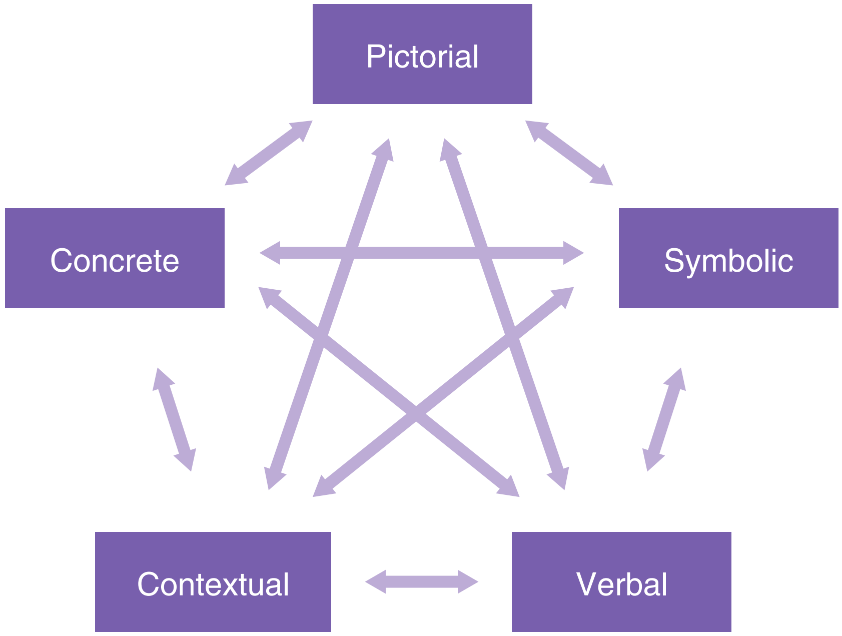
PICTORIAL Pictures and diagrams can illustrate and clarify the details of the actions and relationships in ways that words and even physical representations cannot. Using dots and sticks, bar models, arrows to show action, number lines, and various graphic organizers helps students see and conceptualize the nature of the actions and relationships.

SYMBOLIC Symbols can be operation signs (+, −, ×, ÷), relational signs (=, <, >), variables (typically expressed as x, y, a, b, etc.), or a wide variety of symbols used in middle school and in later mathematics (k, ∞, ϕ, π, etc.). Even though numerals are more familiar, they are also symbols representing values (2, 0.9, $\frac{1}{2}$, 1,000).

There are two things to know about representations that may be surprising. First, mathematics can be shared *only* through representations. As a matter of fact, it is impossible to share a mathematical idea with someone else without sharing it through a representation! If you write an equation, you have produced a *symbolic* representation. If you describe the idea, you have shared a *verbal* representation. Representations are not solely the manipulatives, graphs, pictures, and drawings of a mathematical idea: They are any mode that communicates a mathematical idea between people.

Second, the strength and value of learning to manipulate representations to explore and solve problems is rooted in their relationship to one another. In other words, the more students can learn to move deftly from one representation to another, translating and/or combining them to fully illustrate their understanding of a problem, the deeper will be their understanding of the operations. Figure 1.1 reveals this interdependence. The five modes of representation are all equally important and deeply interconnected, and they work synergistically. In the chapters that follow, you will see how bringing multiple and synergistic representations to the task of problem solving deepens understanding.

FIGURE 1.1 FIVE REPRESENTATIONS: A TRANSLATION MODEL



Source: Adapted from Lesh, Post, and Behr (1987).

Teaching Students to Mathematize

As we discussed earlier, learning to mathematize word problems to arrive at solutions requires time devoted to exploration of different representations with a focus on developing and drawing on a deep understanding of the operations. We recognize that this isn't always easy to achieve in a busy classroom, hence, the appeal of the strategies mentioned at the beginning of the chapter. But what we know from our work with teachers and our review of the research is that, although there are no shortcuts, structuring exploration to focus on actions and relationships is both essential and possible. Doing so requires three things:

1. Teachers draw on their own deep understanding of the operations and their relationship to different word problem situations to plan instruction.
2. Teachers use a model of problem solving that allows for deep exploration.
3. Teachers use a variety of word problems throughout their units and lessons, to introduce a topic and to give examples during instruction, not just as the "challenge" students complete at the end of the chapter.

In this book we address all three.

BUILDING YOUR UNDERSTANDING OF THE OPERATIONS AND RELATED PROBLEM SITUATIONS

The chapters that follow explore the different operations and the various kinds of word problems—or problem situations—that arise within each. To be sure that all the problems and situational contexts your students encounter are addressed, we drew on a number of sources, including the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010), the work done by the Cognitively Guided Instruction projects (Carpenter, Fennema, & Franke, 1996), earlier research, and our own work with teachers to create tables, one for addition and subtraction situations (Figure 1.2) and another for multiplication and division situations (Figure 1.3). Our versions of the problem situation tables represent the language we have found to resonate the most with teachers and students as they make sense of the various problem types, while still accommodating the most comprehensive list of categories. These tables also appear in the Appendix at the end of the book.

FIGURE 1.2 ADDITION AND SUBTRACTION PROBLEM SITUATIONS

ACTIVE SITUATIONS

	Result Unknown	Change Addend Unknown	Start Addend Unknown	
Add-To	Paulo paid $4.53 for his sandwich. Then he paid $1.50 for a carton of milk to finish his lunch. How much was his lunch? $4.53 + 1.5 = x$ $4.53 = x - 1.5$	Paulo paid $4.53 for the sandwich in his lunch. Then he added a carton of milk to his tray to finish his lunch. The total for his lunch is $6.03. How much is a carton of milk? $4.53 + x = 6.03$ $4.53 = 6.03 - x$	Paulo added a sandwich to his tray. He added a carton of milk that cost $1.50 to his tray. With the sandwich and milk, his lunch cost $6.03. How much does the sandwich cost? $x + 1.5 = 6.03$ $6.03 - 1.5 = x$	
Take-From	There are 186 students in the 7th grade. 35 left to get ready to play in the band at the assembly. How many students are not in the band? $186 - 35 = x$ $35 + x = 186$	There are 186 students in the 7th grade. After the band students left class for the assembly, there were 151 students still in their classrooms. How many students are in the band? $186 - x = 151$ $x + 151 = 186$	35 band students left class to get ready to play in the assembly. There were 151 students left in the classrooms. How many students are in the 7th grade? $x - 35 = 151$ $35 + 151 = x$	

RELATIONSHIP (NONACTIVE) SITUATIONS

	Total Unknown	One Part Unknown		Both Parts Unknown
Part-Part-Whole	The local ice cream shop asked customers to vote for their favorite new flavor of ice cream. 119 customers preferred mint chocolate chip ice cream. 37 preferred açai berry ice cream. How many customers voted? $119 + 37 = x$ $x - 119 = 37$	The local ice cream shop asked customers which new ice cream flavor they like best. 156 customers voted. 37 customers preferred açai berry ice cream. The rest voted for mint chocolate chip ice cream. How many customers voted for mint chocolate chip ice cream? $37 + x = 156$ $x = 156 - 37$		The local ice cream shop held a vote for their favorite new flavor of ice cream. The options were mint chocolate chip and açai berry ice cream. What are some possible combinations of votes? $x + y = 156$ $156 - x = y$
	Difference Unknown	**Greater Quantity Unknown**	**Lesser Quantity Unknown**	
Additive Comparison	Jessie and Roberto both collect baseball cards. Roberto has 53 cards and Jessie has 71 cards. How many fewer cards does Roberto have than Jessie? $53 + x = 71$ $53 = 71 - x$	Jessie and Roberto both collect baseball cards. Roberto has 53 cards and Jessie has 18 more cards than Roberto. How many baseball cards does Jessie have? $53 + 18 = x$ $x - 18 = 53$	Jessie and Roberto both collect baseball cards. Jessie has 71 cards and Roberto has 18 fewer cards than Jessie. How many baseball cards does Roberto have? $71 - 18 = x$ $x + 18 = 71$	

FIGURE 1.3 MULTIPLICATION AND DIVISION PROBLEM SITUATIONS

ASYMMETRIC (NONMATCHING) FACTORS				
	Product Unknown	**Multiplier (Number of Groups) Unknown**	**Measure (Group Size) Unknown**	
Equal Groups	Mayim has 8 vases to decorate the tables at her party. She cuts a ribbon $1\frac{3}{4}$ feet long to tie a bow around the vase. How many feet of ribbon does she need? $8 \times 1\frac{3}{4} = x$ $x \div 8 = 1\frac{3}{4}$	Mayim has some vases to decorate the tables at her party. She uses $1\frac{3}{4}$ feet of ribbon to tie a bow around each vase. If she uses 14 feet of ribbon, how many vases does she have? $x \times 1\frac{3}{4} = 14$ $x = 14 \div 1\frac{3}{4}$	Mayim uses 14 feet of ribbon to tie bows around the vases that decorate the tables at her party. If there are 8 vases, how many feet of ribbon are used on each vase? $8x = 14$ $14 \div 8 = x$	
	Product Unknown (y)	**(Unit) Rate Unknown (k)**	**Measure Unknown (x)**	
Ratio/Rate	Tom drove 60 miles per hour (on average) for 4 hours. How many miles did he travel? $4 \times 60 = y$ $\frac{y}{4} = 60$	Tom drove at the same speed (on average) during his entire 4 hour trip. He traveled a total of 240 miles. At what speed did he travel? $4k = 240$ $\frac{240}{4} = k$	Tom drove 60 miles per hour (on average) for all 240 miles of his trip. For how many hours did he travel? $60x = 240$ $\frac{240}{x} = 4$	
	Resulting Value Unknown	**Scale Factor (Times as many) Unknown**	**Original Value Unknown**	
Multiplicative Comparison	Armando's family is doing a puzzle this week that has 500 pieces. Next week's puzzle has 1.5 times as many pieces. How many pieces does next week's puzzle have? $500 \times 1.5 = x$ $x \div 1.5 = 500$	Sydney's middle school has 500 students. José's middle school has 750 students. How many times bigger than Sydney's school is José's school? $500x = 750$ $500 = 750 \div x$	Mrs. W didn't order enough tickets for the festival. Mr. D ordered 750 tickets. Mrs. W said, "You bought 1.5 times as many tickets as I did." How many tickets did Mrs. W order? $1.5 \times x = 750$ $750 \div x = 1.5$	
SYMMETRIC (MATCHING) FACTORS				
	Product Unknown	**One Dimension Unknown**		**Both Dimensions Unknown**
Area/Array	Mr. Bradley bought a new mat for the front entrance to the school. One side measured $3\frac{1}{3}$ feet and the other side measured 12 feet. How many square feet does the mat cover? $3\frac{1}{3} \times 12 = x$ $x \div 12 = 3\frac{1}{3}$	The 40 members of the student council lined up on the stage to take yearbook pictures. The first row included 8 students and the rest of the rows did the same. How many rows were there? $8x = 40$ $x = 40 \div 8$		Daniella was designing a foundation using graph paper. She started with 40 squares. How many units long and wide could the foundation be? $x \times y = 40$ $40 \div x = y$
	Sample Space (Total Outcomes) Unknown	**One Factor Unknown**		**Both Factors Unknown**
Combinatorics** (Probability and Cartesian Products)	Karen has 3 shirts and 7 pairs of pants. How many unique outfits can she make? $3 \times 7 = x$ $3 = x \div 7$	Evelyn says she can make 21 unique ice cream sundaes (1 scoop + 1 topping) using just ice cream flavors and toppings. If she has 3 flavors of ice cream, how many toppings does she have? $3y = 21$ $21 \div 3 = x$		Audrey can make 21 different fruit sodas using the soda mixing machine. How many different flavorings and sodas could there be? $xy = 21$ $x = 21 \div y$

(continued)

*Equal Groups problems, in many cases, are special cases of a category that includes all ratio and rate problem situations. Distinguishing between the two categories is often a matter of interpretation. Since the Ratio/Rate category is a critically important piece of the middle school curriculum and beyond, the Ratio/Rate category is given its own row here.
**Combinatorics (Cartesian products) are typically not included in the table of multiplication and division problem situations. Since this is a category of problem situation addressed in middle school mathematics standards, it has been added to this table.

Note: These representations for the problem situations reflect our understanding based on a number of resources. These include the tables in the Common Core State Standards for mathematics (Common Core Standards Initiative, 2010); the problem situations as described in the Cognitively Guided Instruction research (Carpenter, Hiebert, & Moser, 1981), in Heller and Greeno (1979), and in Riley, Greeno, and Heller (1984); and other tools. See the Appendix and the companion website for a more detailed summary of the documents that informed our development of these tables.

These problem structures are seldom if ever identified in middle-grades standards. They are typically addressed in the early elementary grades as students master basic whole number operations, and taken as known from there. Many of the challenges middle-grades students have with word problems may be rooted in a lack of familiarity with the problem structures, so it is helpful for middle school math teachers to understand them and recognize them within a word problem. We open each chapter in this book with a look at the new problem situation structure with positive rational numbers (whole numbers, fractions, and decimals); the second part of each chapter examines the same structure when the full range of values (positive and negative) are included.

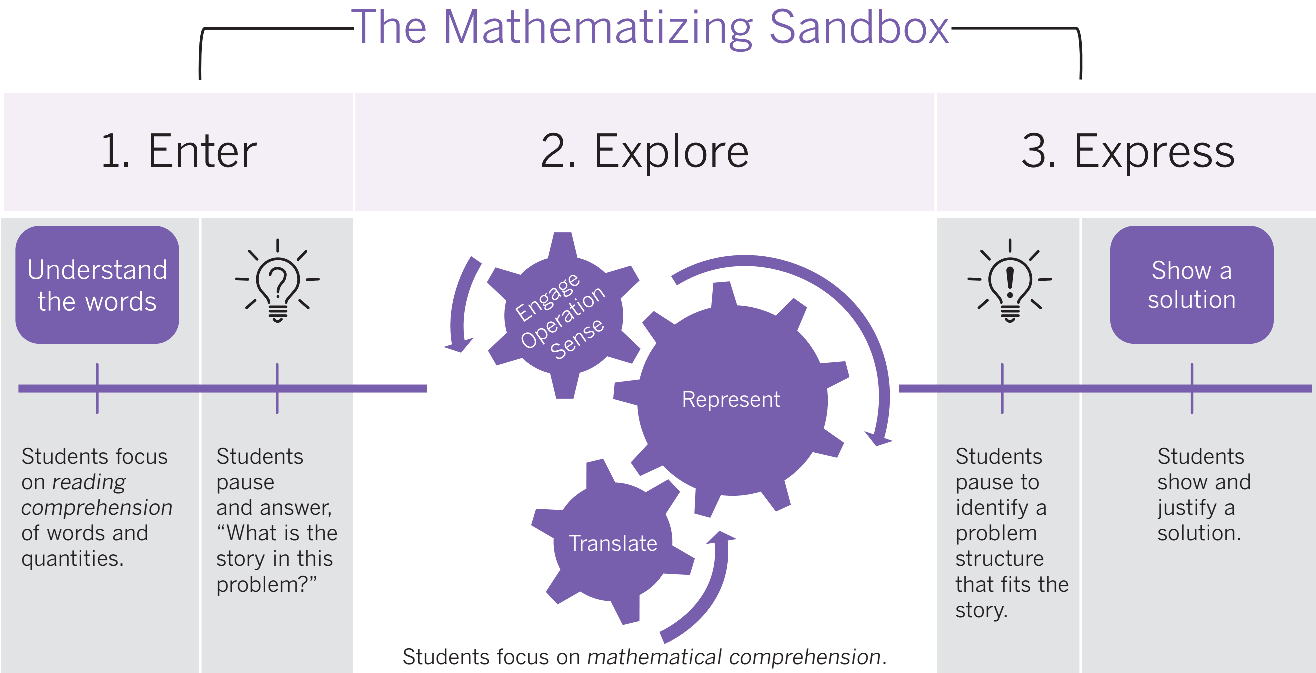
In the chapters—each of which corresponds to a particular problem situation and a row on one of the tables—we walk you through a problem-solving process that enhances your understanding of the operation and its relationship to the problem situation while modeling the kinds of questions and explorations that can be adapted to your instruction and used with your students. Our goal is *not* to have students memorize each of these problem types or learn specific procedures for each one. Rather, our goal is to help you enhance your understanding of the structures and make sure your students are exposed to and become familiar with them. This will support their efforts to solve word problems with understanding—through mathematizing.

In each chapter, you will have opportunities to stop and engage in your own problem solving in the workspace provided. We end each chapter with a summary of the key ideas for that problem situation and some additional practice that can also be translated to your instruction.

EXPLORING IN THE MATHEMATIZING SANDBOX: A PROBLEM-SOLVING MODEL

To guide your instruction and even enhance your own capacities for problem solving, we have developed a model for solving word problems that puts the emphasis squarely on learning to mathematize (Figure 1.4). The centerpiece of this model is what we call the "mathematizing sandbox," and we call it this for a reason. The sandbox is where children explore and learn through play. Exploring, experiencing, and experimenting by using different representations is vital not only to developing a strong operation sense but also to building comfort with the problem-solving process. Sometimes it is messy and slow, and we as teachers need to make room for it. We hope that this model will be your guide.

FIGURE 1.4 A MODEL FOR MATHEMATIZING WORD PROBLEMS



The mathematizing sandbox involves three steps and two pauses:

Step 1 (Enter): Students' first step is one of reading comprehension. Students must understand the words and context involved in the problem before they can really dive into mathematical understanding of the situation, context, quantities, or relationships between quantities in the problem.

Pause 1: This is a crucial moment when, rather than diving into an approach strategy, students make a conscious choice to look at the problem a different way, with a mind toward reasoning and sense-making about the *mathematical story* told by the problem or context. You will notice that we often suggest putting the problem in your own words as a way of making sense. This stage is critical for moving away from the "plucking and plugging" of numbers with no attention to meaning that we so often see (SanGiovanni & Milou, 2018).

Step 2 (Explore): We call this phase of problem solving "stepping into the mathematizing sandbox." This is the space in which students engage their operation sense and play with some of the different representations mentioned earlier, making translations between them to truly understand what is going on in the problem situation. What story is being told? What are we comparing, or what action is happening? What information do we have, and what are we trying to find out? This step is sometimes reflected in mnemonics-based strategies such as STAR (stop, think, act, review) or KWS (What do you know? What do you want to know? Solve it.) or Pólya's (1945) four steps to problem solving (understand, devise a plan, carry out a plan, look back) or even CUBES. But it can't be rushed or treated superficially. Giving adequate space to the Explore phase is essential to the understanding part of any strategic approach. This is where the cognitive sweet spot can be found, and this step is what the bulk of this book is about.

Pause 2: The exploration done in the mathematizing sandbox leads students to the "a-ha moment" when they can match what they see happening in the problem to a known problem situation (Figures 1.2 and 1.3). Understanding the most appropriate problem situation informs which operation(s) to use, but it also does so much more. It builds a solid foundation of operation sense.

Step 3 (Express): Here students leave the sandbox and are ready to express the story either symbolically or even in words, graphs, or pictures, having found a solution they are prepared to discuss and justify.

A Note About Negative Values

Negative rational number values represent multiple challenges for students. The shortcuts and rules that are often taught can feel nonsensical or random, and students may have internalized ideas about computation that are now challenged. For example, students may still believe that addition and multiplication always make things bigger. This is not necessarily true, and that realization is a big cognitive transition for students to make.

We know that integer computation is a challenging skill for many students to develop. It remains, even for some adults, a mystery of mathematics that equations like this one ($^{-}6 - {}^{-}8$) with so many signs expressing a negative value, still yields a positive 2. After all, how can subtraction and two negative numbers possibly yield a positive result? For that matter, why does a negative multiplied by a negative give a positive product? However, our focus in this book is not on computation strategies but, rather, on making sense of problem situations.

We firmly believe that if students reason about the problem situation, they can not only find a solution pathway, but they are more likely to understand where the answer comes from and why it's correct. Further, a deeper understanding of the structure of the problem situations and operations better prepares them to engage in mathematical modeling now as well as in future mathematics classes and into adulthood.

In each chapter, we will explore the problem situation first with fractions, decimals, and whole numbers. In the second half of each chapter, we introduce problem situations that include negative values. We also explore the symbols used in mathematics to describe a negative value. The negative symbol (–) actually has three different meanings (Stephan & Akyuz, 2012):

1. *Subtraction:* This symbol ($-x$), which children learn in elementary school, functions like a verb, an operator between the two values that come before and after the symbol.
2. *Less than zero:* In the middle grades, we introduce a symbol (^{-}x) that distinguishes a negative from a positive number. In this case, the symbol functions more like an adjective. For example, the symbol in front of $^{-}5$ describes a value that is 5 units less than zero. In contrast the symbol in front of $^{+}5$ describes a value 5 units greater than zero.
3. *The opposite:* This use of the negative symbol (^{-}x) conveys the idea of "the opposite," or the additive inverse. In this respect, it toggles back and forth between positive and negative values. Reading ^{-}x as "the opposite of x" instead of as "negative x" communicates that ^{-}x represents the additive inverse of x. If the value of x is already negative, students are often confused by the outcome. For example, when x is $^{-}5$, ^{-}x is the additive inverse of $^{-}5$, or $^{+}5$. How can a number that appears negative (^{-}x) have a positive value ($^{+}5$)?

Distinguishing among these three different uses of the negative symbol may help students recognize them in context and help them be more deliberate in their own use. Conventions about the use of negative numbers are not intuitive for students (Whitacre et al., 2014). They may initially use values and signs (magnitude and direction) in ways that make sense to them but that may or may not correspond to standard conventions (Kidd, 2007). The flashlight problem at the beginning of this chapter is typical. The student's solution relied entirely on positive numbers and a subtraction operator to find the correct answer ($10 - 2 = 8$). This worked for the student likely because she recognized that the explorer never reached 0 to leave the cave. However, a more accurate equation for a problem situation that describes a descent and a climb out of a cave needs to include negative values to be accurate, as in $^{-}10 + x = {}^{-}2$. Does this matter? In this book we will make the case that it does matter. The incorrect equation given by this student may not be so much a "mistake" as it is a mistranslation of her understanding of the problem situation to a more accurate notation. We will return many more times to this idea of connecting the meaning of a problem situation to the various representations used to describe it.

Final Words Before You Dive In

We understand that your real life in a school and in your classroom puts innumerable demands on your time and energy as you work to address ambitious mathematics standards. Who has time to use manipulatives, draw pictures, and spend time writing about mathematics? Your students do! This is what meeting the new ambitious standards actually requires. It may feel like pressure to speed up and do more, but paradoxically, the way to build the knowledge and concepts that are currently described in the standards is by slowing down. Evidence gathered over the past 30 years indicates that an integrated and connected understanding of a wide variety of representations of mathematical ideas is one of the best tools in a student's toolbox (or sandbox!) for a deep and lasting understanding of mathematics (Leinwand et al., 2014). We hope that this book will be a valuable tool as you make or renew your commitment to teaching for greater understanding.